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BOUNDS ON THE NUMBER OF CYCLES OF LENGTH THREE IN A PLANAR GRAPH†

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ABSTRACT

Let G be a p-vertex planar graph having a representation in the plane with nontriangular faces $F_1, F_2, \cdots, F_r$. Let $f_1, f_2, \cdots, f_r$ denote the lengths of the cycles bounding the faces $F_1, F_2, \cdots, F_r$ respectively. Let $C_3(G)$ be the number of cycles of length three in G. We give bounds on $C_3(G)$ in terms of $p, f_1, f_2, \cdots, f_r$. When G is 3-connected these bounds are bounds for the number of triangles in a polyhedron. We also show that all possible values of $C_3(G)$ between the maximum and minimum value are actually achieved.

1. Introduction

Our terminology and notation will be standard throughout this paper. A good reference for undefined terms is [5].

Let G be a planar graph without loops or multiple edges. Denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$, with $p = |V(G)|$ and $q = |E(G)|$. Let $C_3(G)$ denote the number of cycles of length three (for brevity, 3-cycles) in G. In a previous paper [3], the authors proved the following result.

THEOREM 1. *Let G be a maximal planar graph on $p \geqq 6$ vertices. Then*

$$2p - 4 \leqq C_3(G) \leqq 3p - 8.$$

Moreover, for every integer $s \neq 3p - 9$ such that $2p - 4 \leqq s \leqq 3p - 8$, there exists a p-vertex maximal planar graph G with $C_3(G) = s$. There does not exist a p-vertex maximal planar graph G with $C_3(G) = 3p - 9$.

The purpose of this paper is to give bounds for $C_3(G)$ when G is a planar (though not necessarily maximal planar) graph with vertex connectivity $\geqq 2$. If

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such a graph is embedded in the plane, we will call the cycles which bound the faces the *facial cycles* of this embedding, while all remaining cycles will be called the *nonfacial* (or *separating*) *cycles* of the embedding. Let $f_1 \geqq f_2 \geqq \cdots \geqq f_r \geqq 4$ be a sequence of integers. We will say that a graph G satisfies property $P(f_1, f_2, \cdots, f_r)$, if G is 2-connected planar, and if G has an embedding in the plane with exactly r facial cycles which are not 3-cycles, and the lengths of these r facial cycles are $f_1, f_2, \cdots, f_r$. In the following section, we give bounds for $C_3(G)$ if G has property $P(f_1, f_2, \cdots, f_r)$. We then give tighter bounds for $C_3(G)$ under the additional assumption that G is 3-connected. In the final section, we establish that, in contrast to the situation for maximal planar graphs, all possible values of $C_3(G)$ between the maximum and minimum value are actually achieved for some p-vertex graph with property $P(f_1, f_2, \cdots, f_r)$.

2. Basic results

We begin with some preliminary observations. Let G be a p-vertex graph with property $P(f_1, f_2, \cdots, f_r)$ so that in the corresponding embedding there are Δ facial 3-cycles. It follows at once by Euler's formula that $\Delta + r = |E(G)| - p + 2$. Since each edge of G belongs to exactly two facial cycles, we have $2|E(G)| = 3\Delta + \Sigma_{i=1}^r f_i$. From these two equalities, we obtain

$$\Delta = 2p - 4 + 2r - \sum_{i=1}^{r} f_i = 2p - 4 - \sum_{i=1}^{r} (f_i - 2) \tag{1}$$

and

$$|E(G)| = 3p - 6 + 3r - \sum_{i=1}^{r} f_i = 3p - 6 - \sum_{i=1}^{r} (f_i - 3). \tag{2}$$

We now prove the following.

LEMMA 1. *Let $f_1 \geqq f_2 \geqq \cdots \geqq f_r \geqq 4$ be a sequence of integers. Then there exists a p-vertex graph G satisfying property $P(f_1, f_2, \cdots, f_r)$ if and only if $p \geqq f_1$ and $2p - 4 - \Sigma_{i=1}^r (f_i - 2) \geqq 0$. Moreover, in that case G can be constructed so that G contains no separating 3-cycles.*

REMARK. It is worth noting that in its dual form this lemma becomes the problem of realizing a sequence of integers as the vertex degrees of a planar, 2-connected, 3-edge-connected multigraph G'. Moreover, each 3-edge-cut in G' must consist of the edges incident at a degree three vertex. The difficulty in the proof here is primarily due to the above connectivity requirements.

PROOF. If G satisfies property $P(f_1, f_2, \cdots, f_r)$, then by (1) we have $2p - 4 - \sum_{i=1}^{r}(f_i - 2) = \Delta \geqq 0$. Also, since G is 2-connected, the facial cycles are simple cycles and consequently $p \geqq f_1$. This proves the necessity.

To prove the sufficiency, we proceed as follows. Let $j \leqq r$ be the largest integer such that $f_1 + \sum_{i=2}^{j}(f_i - 2) \leqq p$. Since $p \geqq f_1$, it is clear that $j \geqq 1$. Consider the graph G' shown in Fig. 1, where the face F_i is bounded by a cycle of length f_i, for $i = 1, 2, \cdots, j$, and $p' = f_1 + \sum_{i=2}^{j}(f_i - 2)$.

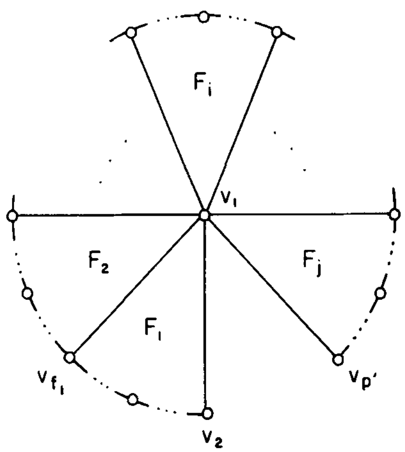

Fig. 1. Graph constructed in the proof of Lemma 1.

If $j = r$, we obtain the desired graph G from G' in two steps as follows:

(a) Consider the exterior face of G' bounded by the cycle $C' = (v_1, v_2, \cdots, v_{p'}, v_1)$. We wish to triangulate this exterior face such that no separating 3-cycles are created in the process. It is easy to see that this can be done unless there exists a vertex u which is adjacent to every vertex v of C' (with $v \neq u$ if u is on C'). Since no such vertex u exists, the desired triangulation can be accomplished.

Let G_1' denote the graph which results when (a) is completed. Then the facial cycles of G_1' which are not 3-cycles have precisely the lengths $f_1, f_2, \cdots, f_r$. Moreover G_1' has $p' \leqq p$ vertices, and is 2-connected with no separating 3-cycles. Hence, if $p' = p$, G_1' itself is the graph we seek. Assuming $p' < p$, we can construct the desired graph G from G_1', by adding $p - p'$ vertices to G_1' as follows:

(b) Select any facial cycle C in G_1' having length $l \geqq 4$. Let x, y, z be any three consecutive vertices on C such that (x, z) is not an edge in G_1'. (It is easily seen that such a choice is always possible.) Add vertex $v_{p'+1}$ inside C and join

$v_{p'+1}$ to x, y, and z. The resulting $(p'+1)$-vertex graph has no separating 3-cycles, and satisfies property $P(f_1, f_2, \cdots, f_r)$. If $p'+1 < p$, repeat the above construction until we obtain a p-vertex graph with no separating 3-cycles having property $P(f_1, f_2, \cdots, f_r)$.

This completes the proof if $j = r$.

Suppose then that $j < r$. We begin by modifying the graph G' in Fig. 1 as shown in Fig. 2, where the face F_{j+1} is created by the addition of $p - p'$ vertices forming a path from $v_{p'}$ to v_p together with an edge (v_p, v_a) such that the length of the cycle bounding F_{j+1} is f_{j+1}. This immediately implies that $a = f_{j+1} + p' - p - 1 \geqq 2$. Denote the resulting graph by G''. We see that G'' is a 2-connected, p-vertex graph having facial cycles of lengths $f_1, f_2, \cdots, f_{j+1}$, with the exterior face bounded by a cycle C of length $p - a + 1 = 2p - 2 - \Sigma_{i=1}^{j+1}(f_i - 2)$. This implies that

$$a = \sum_{i=1}^{j+1}(f_i - 2) - p + 3. \tag{3}$$

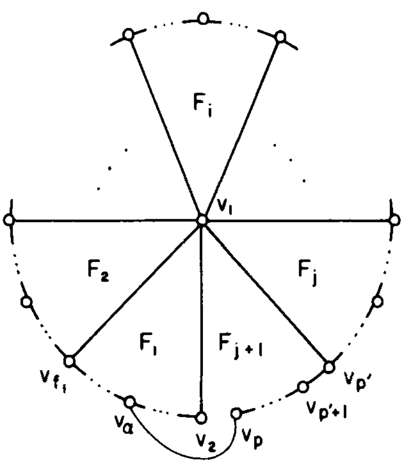

Fig. 2. Graph constructed in the proof of Lemma 1.

We now form facial cycles of lengths $f_{j+2}, f_{j+3}, \cdots, f_r$ by adding edges $(v_a, v_{a_{j+2}}), (v_a, v_{a_{j+3}}), \cdots, (v_a, v_{a_r})$ in the exterior of C, where $a_k = (a+1) + \Sigma_{l=j+2}^{k}(f_l - 2)$, for $k = j+2, j+3, \cdots, r$. We observe that facial cycles of all desired lengths will be present if $a + 1 + \Sigma_{l=j+2}^{r}(f_l - 2) \leqq p$. This inequality, however, is implied by the hypothesis and (3). Finally, we note that if $(a+1) + \Sigma_{l=j+2}^{r}(f_l - 2) = a_r < p$, then there remains an unwanted facial cycle $(v_a, v_{a_r}, v_{a_{r+1}}, \cdots, v_p, v_a)$ of length > 3 which can be triangulated without forming a separating 3-cycle as in (a) above. This yields the desired graph, and completes the proof of Lemma 1.

We now begin considering bounds for $C_3(G)$. The following theorem gives a tight lower bound for $C_3(G)$.

THEOREM 2. *Let G be a p-vertex graph having property* $P(f_1, f_2, \cdots, f_r)$. *Then*

$$C_3(G) \geqq (2p-4) - \sum_{i=1}^{r} (f_i - 2).$$

Furthermore, there exists a p-vertex graph G satisfying property $P(f_1, f_2, \cdots, f_r)$ *for which* $C_3(G) = (2p-4) - \Sigma_{i=1}^{r} (f_i - 2)$.

PROOF. By (1), the number of facial 3-cycles in G is equal to $(2p-4) - \Sigma_{i=1}^{r} (f_i - 2)$, and so $C_3(G) \geqq (2p-4) - \Sigma_{i=1}^{r} (f_i - 2)$. The existence of a p-vertex graph G with property $P(f_1, f_2, \cdots, f_r)$ for which $C_3(G) = (2p-4) - \Sigma_{i=1}^{r} (f_i - 2)$ is implied by Lemma 1. This completes the proof of Theorem 2.

In the remainder of this paper, we will concentrate on giving upper bounds for $C_3(G)$, and determining the quality of these bounds. We begin with the following theorem.

THEOREM 3. *Let G be a p-vertex graph with property* $P(f_1, f_2, \cdots, f_r)$. *Assume* $f_1 \geqq f_2 \geqq \cdots \geqq f_r$. *Set*

$$U_p(f_1, f_2, \cdots, f_r) = \min \left\{ 3p + 4r - 8 - \left\lceil \frac{3}{2} \sum_{i=1}^{r} f_i \right\rceil, \ 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i \right\}.$$

Then

(i) $C_3(G) \leqq U_p(f_1, f_2, \cdots, f_r)$.

(ii) *There exists a p-vertex graph G' with property* $P(f_1, f_2, \cdots, f_r)$ *such that: in case* $f_1 \geqq 3 - r + \lfloor \frac{1}{2} \Sigma_{i=1}^{r} f_i \rfloor$, *then*

$$C_3(G) = U_p(f_1, f_2, \cdots, f_r),$$

and if $f_1 < 3 - r + \lfloor \frac{1}{2} \Sigma_{i=1}^{r} f_i \rfloor$, *then*

$$C_3(G) \geqq U_p(f_1, f_2, \cdots, f_r) - (r-1).$$

(iii) *If* $f_1 \geqq 3 - r + \lfloor \frac{1}{2} \Sigma_{i=1}^{r} f_i \rfloor$, *then* $U_p(f_1, f_2, \cdots, f_r) = 3p + 2r - 4 - f_1 - \Sigma_{i=1}^{r} f_i$, *and for every integer s with* $(2p-4) - \Sigma_{i=1}^{r} (f_i - 2) \leqq s \leqq U_p(f_1, f_2, \cdots, f_r)$, *there exists a p-vertex graph G with property* $P(f_1, f_2, \cdots, f_r)$ *such that* $C_3(G) = s$.

REMARK. The upperbound $U_p(f_1, f_2, \cdots, f_r)$ does not seem tight when $f_1 < 3 - r + \lfloor \frac{1}{2} \Sigma_{i=1}^{r} f_i \rfloor$. Also note that when $f_1 \geqq 3 - r + \lfloor \frac{1}{2} \Sigma_{i=1}^{r} f_i \rfloor$, $C_3(G)$ can be any number in the possible range, in contrast to the result of Theorem 1.

PROOF. (i) We first show that $C_3(G) \leqq 3p + 4r - 8 - \lceil \frac{3}{2}\sum_{i=1}^r f_i \rceil$. Since all variables involved in the inequality are integers, it suffices to show instead that $C_3(G) \leqq 3p + 4r - 8 - \frac{3}{2}\sum_{i=1}^r f_i$. When $r = 0$, this inequality follows by Theorem 1. We therefore proceed by induction on r.

Suppose therefore that $r \geqq 1$ is the smallest integer such that there exists a p-vertex graph G having property $P(f_1, f_2, \cdots, f_r)$ with

$$C_3(G) > 3p + 4r - 8 - \frac{3}{2}\sum_{i=1}^{r} f_i. \tag{4}$$

We may also assume that p is minimal in this regard. We begin by showing that

$$3p + 4r - 8 - \frac{3}{2}\sum_{i=1}^{r} f_i \geqq \Delta = (2p-4) - \sum_{i=1}^{r}(f_i - 2). \tag{5}$$

We may rewrite (5) as

$$p + 2r - 4 - \frac{1}{2}\sum_{i=1}^{r} f_i \geqq 0. \tag{6}$$

Since $\Delta = (2p-4) - \sum_{i=1}^r (f_i - 2) \geqq 0$, we have

$$p + r - 2 - \frac{1}{2}\sum_{i=1}^{r} f_i \geqq 0. \tag{7}$$

Obviously (7) implies the validity of (6) if $r \geqq 2$. We note that $r \geqq 1$ and if $r = 1$, (6) can be rewritten as $p - 2 - \frac{1}{2}f_1 \geqq 0$, which is true since $p \geqq f_1 \geqq 4$. This proves the validity of (5).

From (4) and (5) we conclude that $C_3(G) > \Delta$. This implies that G has at least one separating 3-cycle in this embedding, say $T = (a, b, c, a)$. Let G_1 (resp., G_2) denote the subgraph of G induced by the vertices a, b, c and all vertices of G in the interior (resp, exterior) of T. Note that G_1 and G_2 share only the 3-cycle T, and that each facial cycle of G belongs to either G_1 or G_2. Let $S_1 \cup S_2$ be a partition of $\{1, 2, \cdots, r\}$ into sets (one of which may be empty) such that facial cycles of lengths $\{f_j \mid j \in S_i\}$ belong to G_i, for $i = 1, 2$. Let p_1 and $p_2 (< p)$ be the number of vertices of G_1 and G_2, respectively. Then by the induction hypothesis (and minimality of p)

$$C_3(G_i) \leqq 3p_i + 4|S_i| - 8 - \frac{3}{2}\sum_{j \in S_i} f_j, \tag{8}$$

for $i = 1, 2$.

Clearly $p = p_1 + p_2 - 3$ and $C_3(G) = C_3(G_1) + C_3(G_2) - 1$. Adding the equations in (8) for $i = 1, 2$, and making use of the last two equalities, we obtain

$$(9) \qquad C_3(G) \leqq 3(p_1 + p_2) + 4r - 16 - \frac{3}{2}\sum_{i=1}^{r} f_i - 1$$

or

$$(10) \qquad C_3(G) \leqq 3p + 4r - 8 - \frac{3}{2}\sum_{i=1}^{r} f_i.$$

This contradicts (4), however. We conclude therefore that indeed

$$(11) \qquad C_3(G) \leqq 3p + 4r - 8 - \frac{3}{2}\sum_{i=1}^{r} f_i$$

as desired.

We next want to prove that

$$(12) \qquad C_3(G) \leqq 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i$$

when $r \geqq 1$.

We begin by proving (12) when $r = 1$; which is, that $C_3(G) \leqq 3p - 2 - 2f_1$. Suppose otherwise. Let G be a p-vertex graph (with p minimal) having property $P(f_1)$ such that

$$(13) \qquad C_3(G) > 3p - 2 - 2f_1.$$

Using analogous reasoning as in the first part of the proof, we can prove that G has a separating 3-cycle $T = (a, b, c, a)$. Let G_1 and G_2 be defined as above. We may assume, without loss of generality, that G_1 contains the facial cycle of size f_1, and that G_2 is maximal planar. Then

$$C_3(G_1) = C_3(G) - C_3(G_2) + 1 > 3p - 2 - 2f_1 - (3p_2 - 8),$$

or

$$(14) \qquad C_3(G_1) > 3p_1 - 2 - 2f_1.$$

But this violates the minimality of p. We have thus proved (12) when $r = 1$.

We proceed therefore by induction on r. Assume that $r > 1$ is the smallest integer such that there exists a p-vertex graph G with property $P(f_1, f_2, \cdots, f_r)$ with

$$(15) \qquad C_3(G) > 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i$$

and assume moreover that p is minimal in this regard. It is easily seen that $3p + 2r - 4 - f_1 - \Sigma_{i=1}^{r} f_i \geqq \Delta$, and hence G contains a separating 3-cycle, say

$T=(a,b,c,a)$. Define subgraphs G_1, G_2 of G as before, and suppose without loss of generality that G_1 has a facial cycle of length f_1. If G_2 is maximal planar, then $C_3(G_2) \leqq 3p_2 - 8$. This fact together with (15) yields

$$C_3(G_1) = C_3(G) - C_3(G_2) + 1 > \left(3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i\right) - (3p_2 - 8) + 1,$$

or

$$C_3(G_1) > 3p_1 + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i.$$

This violates the minimality of p. Hence we may assume that G_2 contains at least one facial cycle of length $\geqq 4$. Let $S_1 \cup S_2$ be a partition of $\{1,2,\cdots,r\}$ into nonempty sets such that facial cycles of lengths $\{f_j \mid j \in S_i\}$ belong to G_i, for $i = 1,2$. By the induction hypothesis, we have

$$C_3(G_1) \leqq 3p_1 + 2|S_1| - 4 - f_1 - \sum_{j \in S_1} f_j \tag{16}$$

and

$$C_3(G_2) \leqq 3p_2 + 2|S_2| - 4 - \max_{j \in S_2} f_j - \sum_{j \in S_2} f_j. \tag{17}$$

Adding (16) and (17), and using the fact that $\max_{j \in S_2} f_j \geqq 4$, we obtain

$$C_3(G) = C_3(G_1) + C_3(G_2) - 1 \leqq 3(p_1 + p_2) + 2r - 12 - f_1 - \sum_{i=1}^{r} f_i - 1$$

$$= 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i.$$

This contradicts (15), and establishes the validity of (12). The proof of (i) is now complete.

(ii) Let p_0 be the smallest positive integer such that $p_0 \geqq f_1$ and $2p_0 + 2r - 4 - \Sigma_{i=1}^{r} f_i \geqq 1$. Then[†]

$$p_0 = \max\left\{f_1, \left\lceil \frac{1}{2}\left(5 - 2r + \sum_{i=1}^{r} f_i\right)\right\rceil\right\} = \max\left\{f_1, 3 - r + \left\lfloor \frac{1}{2}\sum_{i=1}^{r} f_i \right\rfloor\right\}. \tag{18}$$

So by Lemma 1, there exists a graph G_0 with p_0 vertices with property $p(f_1, f_2, \cdots, f_r)$ such that $C_3(G_0) = 2p_0 + 2r - 4 - \Sigma_{i=1}^{r} f_i = \Delta$ and $\Delta \geqq 1$. We then construct G from G_0 by recursively adding $p - p_0$ vertices in the interior of a

[†] If $p_0 > p$, then $2p + 2r - 4 - \Sigma_{i=1}^{r} f_i = 0$ and $f_1 < 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$. Part (ii) of Theorem 3 is trivially correct as $U_p(f_1, f_2, \cdots, f_r) - (r-1) < 0$. Therefore, it is assumed that $p_0 \leqq p$.

facial 3-cycle, and joining each vertex to the three vertices incident to the face. Note that the addition of each such vertex creates three additional 3-cycles. As a result, $C_3(G) = C_3(G_0) + 3(p - p_0)$, or

$$C_3(G) = 3p - p_0 + 2r - 4 - \sum_{i=1}^{r} f_i. \tag{19}$$

If $f_1 \geqq 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$, then, by (18), $p_0 = f_1$ and $C_3(G) = 3p + 2r - 4 - f_1 - \Sigma_{i=1}^{r} f_i$. If on the other hand $f_1 < 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$, then $p_0 = 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$, and so by (19) we have $C_3(G) = 3p + 3r - 7 - \lfloor \frac{3}{2}\Sigma_{i=1}^{r} f_i \rfloor$. We summarize this as follows:

$$C_3(G) = \begin{cases} 3p + 3r - 7 - \left\lfloor \frac{3}{2} \sum_{i=1}^{r} f_i \right\rfloor, & \text{if } f_1 < 3 - r + \left\lfloor \frac{1}{2} \sum_{i=1}^{r} f_i \right\rfloor, \\ 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i, & \text{if } f_1 \geqq 3 - r + \left\lfloor \frac{1}{2} \sum_{i=1}^{r} f_i \right\rfloor. \end{cases} \tag{20}$$

It is easily seen that if $f_1 < 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$, then $U_p(f_1, f_2, \cdots, f_r) \leqq 3p + 4r - 8 - \lceil \frac{3}{2}\Sigma_{i=1}^{r} f_i \rceil$, and consequently $C_3(G) \geqq U_p(f_1, f_2, \cdots, f_r) - (r-1)$. On the other hand, if $f_1 \geqq 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$, then $U_p(f_1, f_2, \cdots, f_r) = 3p + 2r - 4 - f_1 - \Sigma_{i=1}^{r} f_i$, and consequently $U_p(f_1, f_2, \cdots, f_r) = C_3(G)$. This completes the proof of (ii).

(iii) Suppose that $f_1 \geqq 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$. We have already established that if G is a p-vertex graph with property $P(f_1, f_2, \cdots, f_r)$, then

$$2p - 4 - \sum_{i=1}^{r} (f_i - 2) \leqq C_3(G) \leqq 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i. \tag{21}$$

We now show that for any integer s in the range implied by (21), there exists a p-vertex graph G having property $P(f_1, f_2, \cdots, f_r)$ such that $C_3(G) = s$.

Since $f_1 \geqq 3 - r + \lfloor \frac{1}{2}\Sigma_{i=1}^{r} f_i \rfloor$, (18) implies that $f_1 = p_0$. So by Lemma 1, a p_0-vertex graph G_0 can be constructed for which $C_3(G_0) = 2p_0 + 2r - 4 - \Sigma_{i=1}^{r} f_i = \Delta \geqq 1$. To construct G from G_0, we add $p - p_0$ vertices to G_0 as follows: Let $0 \leqq x \leqq p - p_0$. We add x vertices to G_0 in the manner described in Lemma 1 (i.e., each vertex is placed in the interior of a facial cycle of length $\geqq 4$, and then joined to three consecutive vertices on the cycle). Once again, this can be accomplished without creating separating 3-cycles. Once these x vertices are added, the resulting $(p_0 + x)$-vertex graph G_1 satisfies property $P(f_1, f_2, \cdots, f_r)$ and contains $C_3(G_0) + 2x$ 3-cycles. Next, we add $p - p_0 - x$ vertices to G_1 by recursively placing a vertex in the interior of a facial 3-cycle and joining this vertex to the three vertices incident to the 3-cycle. The resulting graph G is a

p-vertex graph with property $P(f_1, f_2, \cdots, f_r)$, and also $C_3(G) = C_3(G_0) + 2x + 3(p - p_0 - x)$, or

$$C_3(G) = 3p - p_0 - x + 2r - 4 - \sum_{i=1}^{r} f_i. \tag{22}$$

Since x was any integer such that $0 \leqq x \leqq p - p_0$, it follows that $C_3(G)$ can assume any value from $2p + 2r - 4 - \Sigma_{i=1}^{r} f_i$ to $3p - p_0 + 2r - 4 - \Sigma_{i=1}^{r} f_i$. Noting that $p_0 = f_1$, we have the desired result.

The proof of Theorem 3 is complete.

We now wish to show that Theorem 3 provides nearly complete results when $r = 1$ or 2. More precisely, we have the following result.

COROLLARY. *Let G be a p-vertex graph satisfying property $P(f_1, f_2, \cdots, f_r)$. Assume $f_1 \geqq f_2 \geqq \cdots \geqq f_r$, and $1 \leqq r \leqq 2$, with $f_1 > f_2$ if $r = 2$. Then*

$$2p - 4 - \sum_{i=1}^{r} (f_i - 2) \leqq C_3(G) \leqq 3p + 2r - 4 - f_1 - \sum_{i=1}^{r} f_i.$$

Moreover for every integer s in the range indicated above, there exists a p-vertex graph G satisfying property $P(f_1, f_2, \cdots, f_r)$ such that $C_3(G) = s$.

PROOF. The validity of the bounds are established in Theorem 3. To prove the second part of the corollary, note that if $r = 1$, then, since $f_1 \geqq 3 - 1 + \lfloor \frac{1}{2} f_1 \rfloor$, Theorem 3 (iii) gives the desired result. Also if $r = 2$ and $f_1 \geqq f_2 + 1$, then $f_1 \geqq 3 - 2 + \lfloor \frac{1}{2}(f_1 + f_2) \rfloor$, and by Theorem 3 (iii), the corollary follows.

We noted that the upper bound for $C_3(G)$ in Theorem 3 is not uniformly tight when $f_1 < 3 - r + \lfloor \frac{1}{2} \Sigma_{i=1}^{r} f_i \rfloor$. Nevertheless, there are graphs for which this upper bound is achieved. For instance, let $p = 9$, $r = 5$, $f_1 = 5$, and $f_2 = f_3 = f_4 = f_5 = 4$. Note that then $U_9(5,4,4,4,4) = 7$. We observe that the graph G of Fig. 3 satisfies property $P(5,4,4,4,4)$ and has $C_3(G) = 7$. Such occurrences where the upper bound is attained, though not unique, are by no means common either. In fact, they probably can occur either when $f_2 = f_1 - 1$, and $f_3 = f_4 = \cdots = f_r = 4$, or when $p > r + 2$ and $f_1 = f_2 = \cdots = f_r = 4$. On the other hand, let $p = 23$, $r = 10$,

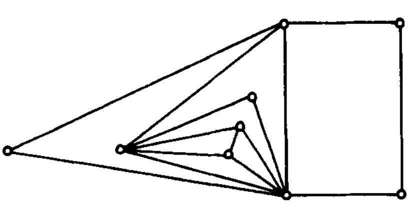

Fig. 3. A graph which achieves the upperbound in Theorem 3.

$f_1 = 16$, $f_2 = 8$, and $f_3 = f_4 = \cdots = f_{10} = 6$. Note that $U_{33}(16, 8, 6, 6, \cdots, 6) = \min\{23, 27\} = 23$. The 33-vertex graph G shown in Fig. 4 is a 2-connected, has the appropriate facial structure, and is believed to have the largest possible number of 3-cycles subject to these restrictions and yet $C_3(G) = 17$. It should be noted, however, that $C_3(G)$ still differs from $U_{33}(16, 8, 6, \cdots, 6)$ by less than the quantity $r - 1 = 9$ as predicted in Theorem 3. The essential difficulty in finding an achievable upperbound stems from our inability to control the number of separating 3-cycles in the p_0-vertex graph that is constructed, where p_0 is given by (18).

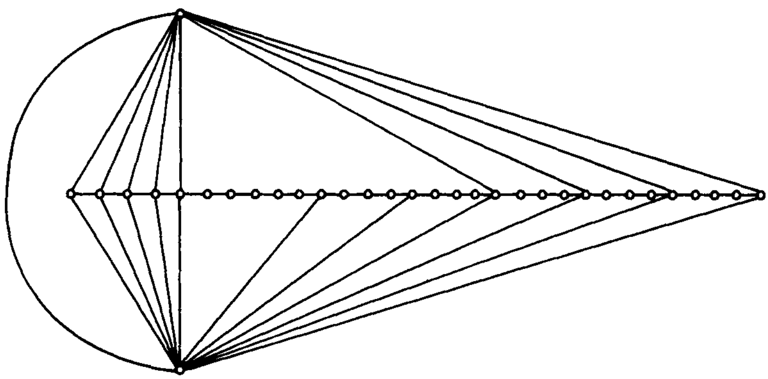

Fig. 4. A graph which is believed to have the largest number of 3-cycles and yet does not achieve the bound of Theorem 3.

3. Bounds on $C_3(G)$ when G is 3-connected

In a planar graph G with connectivity 2 (such as the graphs in Figs. 3 and 4), the lengths of the facial cycles depend on the particular embedding of the graph G in the plane. On the other hand, it is known (see [5, p. 105], [7]) that if a planar graph G has connectivity $\geqq 3$, then G has a unique embedding in the plane, and hence the lengths of the facial cycles are uniquely determined. Moreover, it is known (see [1, 2]) that a graph G will be the 1-skeleton of a polyhedron if and only if G is 3-connected and planar. This means that the study of the number of 3-cycles in polyhedra is equivalent to studying the number of 3-cycles in 3-connected planar graphs.

To begin with, let us consider again the 33-vertex graph G in Fig. 4. Note that $C_3(G) = 17$. By way of contrast, we exhibit in Fig. 5 a 3-connected, 33-vertex planar graph G' whose facial cycle lengths are exactly the same as those in G, but with $C_3(G') = 11$. Our next result shows that G' has the maximum number of

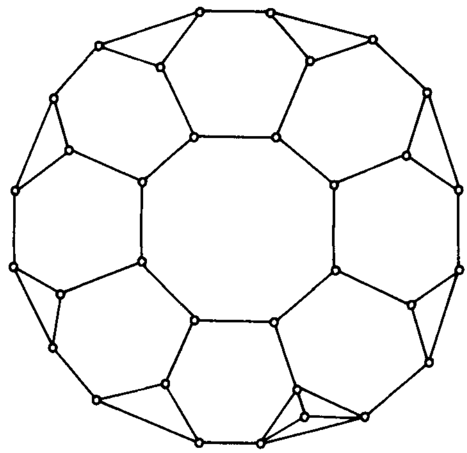

Fig. 5. A 3-connected graph with the same facial cycles and number of vertices as the graph of Fig. 4.

3-cycles among all 3-connected, 33-vertex planar graphs with this facial cycle length.

THEOREM 4. *Let G be a 3-connected, p-vertex graph with property $P(f_1, f_2, \cdots, f_r)$. Then*

$$C_3(G) \leqq 3p + 4r - 8 - \left\lceil \frac{5}{3} \sum_{i=1}^{r} f_i \right\rceil . \tag{23}$$

PROOF. The inequality is true if $r = 0$ by Theorem 1, and so we proceed by induction. Suppose then that $r \geqq 1$ is smallest integer such that there exists a 3-connected, p-vertex planar graph G having property $P(f_1, f_2, \cdots, f_r)$ with

$$C_3(G) > 3p + 4r - 8 - \left\lceil \frac{5}{3} \sum_{i=1}^{r} f_i \right\rceil , \tag{24}$$

and with p minimal in this regard.

We wish to show first that G contains a separating triangle. To do this, we need to show that

$$3p + 4r - 8 - \left\lceil \frac{5}{3} \sum_{i=1}^{r} f_i \right\rceil \geqq \Delta = 2p + 2r - 4 - \sum_{i=1}^{r} f_i,$$

which can be rewritten as

$$3p + 6r - 12 - 2 \cdot \sum_{i=1}^{r} f_i \geqq 0. \tag{25}$$

Since each vertex of G has degree at least three, we have

$$2|E(G)| = \sum_{i=1}^{r} f_i + 3\Delta \geqq 3p. \tag{26}$$

Substituting $2p + 2r - 4 - \Sigma_{i=1}^{r} f_i$ for Δ in (26) yields (25).

If we denote the separating 3-cycle by $T = (a, b, c, a)$ and define G_1 and G_2 as before, it is easy to see that as G is 3-connected so are G_1 and G_2, and hence G_1 and G_2 satisfy the appropriate inequality (23). Arguing now as we have a number of times previously, we would obtain an inequality contradicting (24).

This completes the proof of Theorem 4.

We have observed that the graph in Fig. 5 achieves the upperbound given in Theorem 4. However, in general this bound does not seem tight. For example, if G is a 3-connected, 9-vertex planar graph with property $P(5,4,4,4,4)$, then (23) suggests that $C_3(G) \leqq 4$. In fact, it seems that such a graph will actually satisfy $C_3(G) \leqq 3$, with equality occurring, for example, in the graph of Fig. 6.

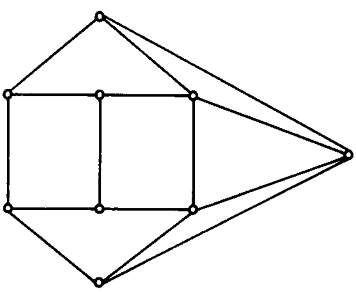

Fig. 6. A 3-connected 9-vertex planar graph with property $P(5,4,4,4,4)$ which is believed to have the largest number of 3-cycles.

We observe that the problem of determining when there exists a p-vertex 3-connected planar graph G with property $P(f_1, f_2, \cdots, f_r)$, regardless of the number of separating 3-cycles (G must always contain $\Delta = 2p - 4 - \Sigma_{i=1}^{r}(f_i - 2)$ facial 3-cycles), is equivalent to a singularly difficult problem in graph theory. To see why, note that the dual of G would be a 3-connected planar graph G_1 with vertex degree sequence $f_1, f_2, \cdots, f_r \underbrace{3, \cdots, 3}_{\Delta}$. Moreover, as p varies Δ will take on all nonnegative odd or even integer values (depending on the parity of $\Sigma_{i=1}^{r} f_i$). Thus we see that knowing when the graph G exists is equivalent in general to knowing when a sequence of integers is the degree sequence of a planar graph without multiple edges. It is known (see [6]) that this problem is exceedingly difficult. This problem is of course much easier if we allow the realization to be a multigraph; in fact, the solution to that problem (in dual form) is precisely the result of Lemma 1. On the other hand, neither the result of Lemma 1 nor a complete solution to the planar degree sequence problem in full generality (if it were possible) would completely settle the problem of finding tight bounds for $C_3(G)$, since we still would have no way of determining or controlling the number of separating 3-cycles in G.

We now give a still tighter bound for $C_3(G)$ than the bound in Theorem 4 when G is a 3-connected planar graph with an additional restriction. A p-vertex graph G is said to satisfy property $P_1(f_1, f_2, \cdots, f_r)$ if (1) G satisfies $P(f_1, f_2, \cdots, f_r)$ and (2) if V_i is the set of vertices on the facial cycle of length f_i, bounding the face F_i, then for every subset $S \subseteq \{1, 2, \cdots, r\}$ of cardinality $\geqq 2$, we have[†]

$$\left| \bigcup_{i \in S} V_i \right| \geqq 4 + \sum_{i \in S} (f_i - 3). \tag{27}$$

We now prove the following.

THEOREM 5. *Let G be a 3-connected p-vertex graph with property $P_1(f_1, f_2, \cdots, f_r)$. Then*

$$2p - 4 - \sum_{i=1}^{r} (f_i - 2) \leqq C_3(G) \leqq 3p - r - 8 - 2 \sum_{i=1}^{r} (f_i - 3).$$

Moreover, if $p \geqq 3 + \sum_{i=1}^{r} (f_i - 2)$ and $r \geqq 1$, and if s is any integer such that

$$2p - 4 - \sum_{i=1}^{r} (f_i - 2) \leqq s \leqq 3p - r - 8 - 2 \sum_{i=1}^{r} (f_i - 3), \tag{28}$$

then there exists a 3-connected p-vertex graph G with property $P_1(f_1, f_2, \cdots, f_r)$ such that $C_3(G) = s$.

PROOF. The lower bound in (28) was established in Theorem 2.

We will prove the upper bound in (28) by induction on r. It is true for $r = 0$ by Theorem 1. We will also need to show it is true for $r = 1$. Suppose therefore that $r = 1$, and that there exists a 3-connected p-vertex graph G such that

$$C_3(G) > 3p - 1 - 8 - 2(f_1 - 3) = 3p - 3 - 2f_1, \tag{29}$$

and assume that p is minimal in this regard. We observe that $3p - 3 - 2f_1 \geqq \Delta = 2p - 2 - f_1$, if $p \geqq f_1 + 1$. To see that $p \geqq f_1 + 1$, we note that if $p = f_1$, then it is easily seen that G would have connectivity 2. It follows at once that $C_3(G) > \Delta$, and hence G has a separating 3-cycle, say $T = (a, b, c, a)$. We define G_1 and G_2

[†] Roughly speaking, (27) states that the cycles bounding the faces, $F_i, i \in S$, do not share too many vertices.

as before, and assume G_1 contains the facial cycle of length f_1. We observe that G_1 and G_2 are both 3-connected, and hence by the assumption that p is minimal, we have $C_3(G_1) \leqq 3p_1 - 3 - 2f_1$ and $C_3(G_2) \leqq 3p_2 - 8$. These two inequalities yield an inequality which violates (29). We conclude the upper bound in (28) is correct when $r = 1$.

We now proceed by induction on r. Suppose that $r \geqq 2$ is the smallest integer such that there exists a 3-connected p-vertex graph G satisfying $P_1(f_1, f_2, \cdots, f_r)$ with

$$C_3(G) > 3p - r - 8 - 2\sum_{i=1}^{r}(f_i - 3), \tag{30}$$

and with p minimal in this regard. We claim that

$$3p - r - 8 - 2\sum_{i=1}^{r}(f_i - 3) \geqq \Delta = 2p - 4 - \sum_{i=1}^{r}(f_i - 2). \tag{31}$$

But (31) can be rewritten as

$$p \geqq 4 + \sum_{i=1}^{r}(f_i - 3) \tag{32}$$

which is obviously implied by condition (27) when $S = \{1, 2, \cdots, r\}$ and $r \geqq 2$. From (31), we conclude that G contains a separating 3-cycle, say $T = (a, b, c, a)$. Let G_1 and G_2 be defined as previously. Let $S_1 \cup S_2$ be a partition of $\{1, 2, \cdots, r\}$ such that the facial cycles of lengths $\{f_j \mid j \in S_i\}$ belong to G_i. It is easy to see that G_i is 3-connected and satisfies property $P_1(\{f_j \mid j \in S_i\})$. So by the induction hypothesis

$$C_3(G_i) \leqq 3p_i - |S_i| - 8 - 2\sum_{j \in S_i}(f_j - 3), \qquad \text{for } i = 1, 2.$$

As before, these two inequalities will lead to a contradiction to (30). This establishes the upperbound in (28).

To prove the second part of Theorem 5, we begin by considering the graph G' in Fig. 7. We note that $p' = |V(G')| = \sum_{i=1}^{r} f_i - 2(r-1) + 1$, and consequently $\Delta(G') = \sum_{i=1}^{r} f_i - 2r + 2$. It can be verified that the number of separating 3-cycles in G' is precisely $r - k$. Therefore, we have

$$C_3(G') = \sum_{i=1}^{r} f_i - r - k + 2, \qquad \text{where } 1 \leqq k \leqq r. \tag{33}$$

We now add $p - p' \geqq 0$ vertices to G' in the manner described previously: x of them are added to create two additional 3-cycles each, and the remaining

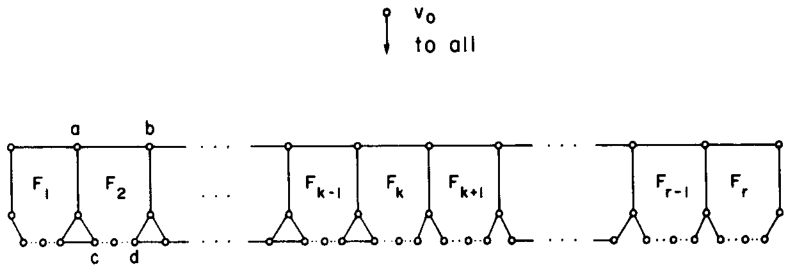

Fig. 7. Graph G' used in the proof of Theorem 5. The arrow emanating from v_0 indicates the existence of edges between v_0 and all "outer" vertices. If, for example, $f_2 = 4$, then it is assumed that vertices a and b are the same as well as the vertices c and d.

$p - p' - x$ are added to create three additional 3-cycles each. Call the resulting graph G. It is easy to see that G is a 3-connected p-vertex graph with property $P_1(f_1, \cdots, f_r)$ with

$$C_3(G) = \sum_{i=1}^{r} f_i = r - k + 2 + 2x + 3(p - p' - x)$$

(34)

$$= 3p + 5r - 7 - 2\sum_{i=1}^{r} f_i - k - x.$$

We observe that as k varies from 1 to r and x varies from 0 to $p - p' = p - \Sigma_{i=1}^{r} f_i + 2r - 3$, then $C_3(G)$ can assume any value s such that

$$2p - 4 - \sum_{i=1}^{r} (f_i - 2) \leqq s \leqq 3p + 5r - 8 - 2\sum_{i=1}^{r} f_i,$$

as asserted. This completes the proof of Theorem 5.

When $r = 1$ or 2, Theorem 5 provides complete results. Because if $r = 1$ condition (27) does not apply and the inequality $p \geqq 3 + (f_1 - 2)$ is necessary for 3-connectivity. If $r = 2$, condition (27) always holds since it is necessary for 3-connectivity; however the hypothesis $p \geqq 3 + \Sigma_{i=1}^{2} (f_i - 2) = f_1 + f_2 - 1$ may not be satisfied. But certainly $p \geqq f_1 + f_2 - 2$ by the 3-connectivity of G. If $p = f_1 + f_2 - 2$, then it can be shown that the upper and lower bounds in Theorem 5 are equal yielding $C_3(G) = \Delta(G) = f_1 + f_2 - 4$, and there is a 3-connected planar graph on $f_1 + f_2 - 2$ vertices satisfying $P_1(f_1, f_2)$ having exactly $f_1 + f_2 - 4$ 3-cycles.

When $r = 3$, there are 3-connected planar graphs for which condition (27) is not fulfilled. An example of such a graph is given in Fig. 8. We also note that this graph does not satisfy the upper bound in (28).

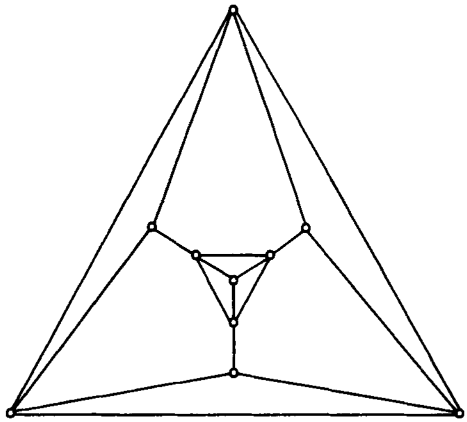

Fig. 8. A 3-connected graph which does not satisfy (27).

4. The range of values of $C_3(G)$

Consider a sequence of integers $f_1 \geqq f_2 \geqq \cdots \geqq f_r \geqq 4$, with $r \geqq 1$. In this section, we will show that given any integer s between the maximum and minimum values of $C_3(G)$ for p-vertex graphs with property $P(f_1, f_2, \cdots, f_r)$, there exists a p-vertex graph G with property $P(f_1, f_2, \cdots, f_r)$ such that $C_3(G) = s$. In contrast to this phenomenon, we have the result of Theorem 1 which shows that for *maximal* planar graphs the complete range of values of $C_3(G)$ is not attainable. In particular, there is no p-vertex maximal planar graph G with $C_3(G) = 3p - 9$.

We begin by showing that the tight lower bound for $C_3(G)$ given in Theorem 2 remains tight if we restrict our attention to 3-connected graphs. More precisely, we have the following lemma.

LEMMA 2. *Suppose there exists a p-vertex, 3-connected graph with property $P(f_1, f_2, \cdots, f_r)$, with $r \geqq 1$. Then there exists a p-vertex, 3-connected graph G with property $P(f_1, f_2, \cdots, f_r)$ such that $C_3(G) = \Delta(G) = (2p - 4) - \Sigma_{i=1}^{r}(f_i - 2)$.*

PROOF. Let G_1 be a p-vertex, 3-connected graph with property $P(f_1, f_2, \cdots, f_r)$ having the least number of 3-cycles. If $C_3(G_1) = \Delta(G_1)$, the lemma is proved. Hence we assume $C_3(G_1) > \Delta(G_1)$, implying that G_1 contains a separating 3-cycle $T = (x, y, z, x)$.

Consider a plane embedding of G_1, and let $G_1' = G_1 - (x, y)$. Let C_{xy} be the cycle bounding the face of G_1' in which the edge (x, y) originally lay. Since G is 3-connected, it is easily verified that C_{xy} does not pass through z, and that G_1' is

3-connected. Let $k \geqq 4$ denote the length of the cycle C_{xy}. Set $x_0 = x$, and denote the vertices on C_{xy} (in clockwise order) by $x_0, x_1, \cdots, x_{k-1}$, with, say, $y = x_\alpha$. For each j such that $0 \leqq j \leqq k-1$, define $G(x_j) = G_1' + (x_j, x_{j+\alpha})$, where the addition of the indices is in modulo k. We note that for each j, $G(x_j)$ is a p-vertex, 3-connected graph with property $P(f_1, f_2, \cdots, f_r)$. The lemma will be proved if we can exhibit a p-vertex, 3-connected graph G_2 with property $P(f_1, f_2, \cdots, f_r)$ such that $C_3(G_2) < C_3(G_1)$.

Suppose first that z is not adjacent (in G_1) to all vertices in C_{xy}. Then we claim that there exists an integer $l, 0 \leqq l \leqq k-1$, such that $C_3(G(x_l)) < C_3(G)$ (which would prove the lemma in this case). To prove the claim, suppose otherwise, that is, $C_3(G(x_j)) \geqq C_3(G)$, for $j = 0, 1, \cdots, k-1$. This implies that for every integer j, the edge $(x_j, x_{j+\alpha})$ belongs to a separating 3-cycle in $G(x_j)$. We begin by considering $G(x_1)$. By the planarity of $G(x_1)$, both x_1 and $x_{1+\alpha}$ must be adjacent to z. Applying this argument recursively, we conclude that z is adjacent to every vertex on C_{xy}, a contradiction. Hence, the lemma is true if z is not adjacent to some vertex on C_{xy}.

Let us assume, therefore, that z is adjacent to every vertex on C_{xy}. If we repeat the above argument with respect to the edge (y, z) in T, we see that the lemma is correct unless x is adjacent to every vertex on C_{yz}. Repeating the argument again for the edge (z, x) in T, we conclude that the lemma is correct unless y is adjacent to every vertex on C_{zx}. Thus the lemma is correct unless x is adjacent to all vertices in C_{yz}, y is adjacent to every vertex in C_{xz}, and z is adjacent to every vertex in C_{xy}. It is easily verified that this can only happen if G is the maximal planar graph on five vertices, and hence is impossible if $r \geqq 1$. The lemma is proved.

Let $\mathcal{G}_k$, where $k = 2$ or 3, denote the set of all p-vertex, k-connected graphs satisfying property $P(f_1, f_2, \cdots, f_r)$. Set $\bar{u}_k(f_1, f_2, \cdots, f_r) = \max_{G \in \mathcal{G}_k} C_3(G)$. We then have the following result.

THEOREM 6. *Let s be any integer such that*

$$2p - 4 - \sum_{i=1}^{r} (f_i - 2) \leqq s \leqq \bar{u}_k(f_1, f_2, \cdots, f_r),$$

for $k = 2, 3$. Then there exists a graph $G \in \mathcal{G}_k$ such that $C_3(G) = s$.

PROOF. We have established the theorem when $r = 1$ (see the corollary to Theorem 3, for $k = 2$; and the discussion following Theorem 5, for $k = 3$). Thus we proceed by induction on r.

Let $G \in \mathcal{G}_k$ be such that $C_3(G) = \bar{u}_k(f_1, f_2, \cdots, f_r)$. If $C_3(G) =$

$2p-4-\Sigma_{i=1}^{r}(f_i-2)=\Delta(G)$, there is nothing to prove. Suppose therefore that $C_3(G)>\Delta(G)$; then G has a separating 3-cycle, say $T=(a,b,c,a)$. As before, let G_1 (resp., G_2) denote the subgraph of G induced by the vertices a, b, and c and all vertices of G in the interior (resp., the exterior) of T. Let G_i have p_i vertices for $i=1,2$, and let $S_1\cup S_2$ be the partition of $\{1,2,\cdots,r\}$ such that $\{f_j \mid j\in S_i\}$ corresponds to the nontriangular faces of G_i, for $i=1,2$. We must have $C_3(G_i)=\bar{u}_k\{f_j \mid j\in S_i\}$ for $i=1,2$, since otherwise $C_3(G)$ would not be equal to $\bar{u}_k(f_1,\cdots,f_r)$. We also note that if G is 3-connected, then so are G_1 and G_2.

Assume for the moment that both S_1 and S_2 are nonempty. Then by the induction hypothesis, we know that for any two integers s_1, s_2 such that

$$2p_1-4-\sum_{j\in S_i}(f_j-2)\leqq s_i\leqq \bar{u}_k\{f_j \mid j\in S_i\} \qquad \text{for } i=1,2,$$

there exists a p_i-vertex graph G_i with property $P(f_j \mid j\in S_i)$ such that $C_3(G_i)=s_i$, for $i=1,2$. Since both of these graphs must have at least one facial 3-cycle, we can "fuse" them together along these 3-cycles to obtain a graph G' with $p=p_1+p_2-3$ vertices, with $C_3(G')=C_3(G_1)+C_3(G_2)-1$, and satisfying property $P(f_1,f_2,\cdots,f_r)$. (We also note that if both G_1 and G_2 are 3-connected, then so is G'.) This immediately implies that for every integer s with $2p-4-\Sigma_{i=1}^{r}(f_i-2)<s\leqq \bar{u}_k(f_1,f_2,\cdots,f_r)$, there exists a graph $G'\in\mathscr{G}_k$ such that $C_3(G')=s$. But we proved the existence of a graph $G''\in\mathscr{G}_k$ with $C_3(G'')=2p-4-\Sigma_{i=1}^{r}(f_i-2)$ in Lemmas 1 and 2. This completes the proof when both S_1 and S_2 are nonempty.

Suppose then that for every separating 3-cycle T in G, either S_1 or S_2 is empty (and hence that G_1 or G_2, respectively, are maximal planar). Without loss of generality, suppose S_1 is empty and thus G_1 is maximal planar. Since $C_3(G)=\bar{u}_k(f_1,\cdots,f_r)$, it follows at once that $C_3(G_1)=3p_1-8$. This implies that G has a vertex v_1 of degree three in the interior of T (see [2]); clearly $C_3(G-v_1)=C_3(G)-3$. Let G^1 be the p-vertex graph obtained from $G-v_1$ by placing a vertex v_1' in the interior of the facial cycle of length f_1 and joining v_1' to three consecutive vertices on the cycle (see (b) in the proof of Lemma 1). Then we have $C_3(G^1)=C_3(G-v_1)+2=C_3(G)-1$. We can continue this process until all p_1-3 vertices of G in the interior of T are exhausted. In this way, we obtain graphs $G^2,G^3,\cdots,G^{p_1-3}$ with $C_3(G^i)=C_3(G)-i$, for $i=1,2,\cdots,p_1-3$. If $C_3(G^{p_1-3})>2p-4-\Sigma_{i=1}^{r}(f_i-2)$, then G^{p_1-3} must have a separating 3-cycle T_1. But any separating 3-cycle in G^{p_1-3} must also be a separating 3-cycle in G. Hence the set of all vertices on and in say the exterior of T_1 will induce a maximal planar graph in G^{p_1-3}. We can thus repeat the above argument, and continue repeating it, to obtain a graph $G \in \mathcal{G}_k$ with $C_3(G)$ being any number in the indicated range.

The proof of Theorem 6 is complete.

The results here may be applied to find bounds for the number of triangles in a triangulation of the plane on a given set of points [4].

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